Tentamen Numerieke Wiskunde 2
14 november 2001

Duration: 3 hours

N.B. Unless stated otherwise, the notation used is as in the book of Burden and Faires.

Problem 1

Consider the ordinary differential equation: $y' = f(t, y)$.

a. Show how the explicit midpoint method given by

$$w_{i+1} = w_{i-1} + 2hf(t_i, w_i)$$

can be derived from an integral equation related to the ordinary differential equation. Use this to show that the method has local truncation error $O(h^2)$.

b. Describe the root condition. Does the explicit midpoint method satisfy this condition? What is the consequence if a method satisfies the root condition?

c. The characteristic polynomial related to the explicit midpoint method has two roots $\beta_k(h\lambda)$, $k = 1, 2$. Given that at least one of these roots is in magnitude larger than one for $h\lambda$ away from the imaginary axes, show that $|\beta_1(h\lambda)| = |\beta_2(h\lambda)| = 1$ for $h\lambda$ on the interval $[-i, i]$ on the imaginary axes.

Problem 2

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$$

for x on $[0, 1]$ and $t \geq 0$ with $u(0, t) = u(1, t) = 0$ and appropriate initial conditions.

a. Show that a semi-discretization (discretization in space only) of the wave equation is given by the system of ordinary differential equations

$$\frac{d^2 v}{dt^2} = Bv,$$

where

$$\begin{aligned} (Bv)_1 &= 4(v_2 - 2v_1)/h^2, \\ (Bv)_j &= 4(v_{j+1} - 2v_j + v_{j-1})/h^2 \text{ for } j = 2, ..., m-2, \\ (Bv)_{m-1} &= 4(-2v_{m-1} + v_{m-2})/h^2 \end{aligned}$$

with $h = 1/m$, and m a natural number. What is the local truncation error?

b. Locate the eigenvalues of the matrix B defined in part a by Gerschgorin's theorem.

c. Show that the system in part a can be converted to the first-order system of ordinary differential equations

$$\frac{dy}{dt} = Cy$$

with

$$C = \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}$$

d. Show that the eigenvalues of C defined in part c are located on the interval $[-4i/h, 4i/h]$ on the imaginary axis. (hint: there are basically two ways: (i) show that C is similar to a skew-symmetric matrix, or (ii) use Gaussian elimination on the eigenvalue equation $(C - \lambda I)x = 0$ assuming that λ is nonzero.)

e. Can the explicit midpoint method given in part a of Problem 1 be applied to this system and, if yes, what is the restriction on the time step?

Problem 3

Consider the system $Ax = b$, with b known and A a tridiagonal matrix of order n with on the main diagonal the value 3, and on the lower and upper diagonal the value -1. So for $n = 4$

$$A = \begin{bmatrix} 3 & -1 & & \\ -1 & 3 & -1 & \\ & -1 & 3 & -1 \\ & & -1 & 3 \end{bmatrix}$$

a. Show that A is strictly diagonally dominant.

b. Use Gershgorin's theorem to estimate the condition number of A.

c. Let $A^{(0)} = A$ and $A^{(1)}, A^{(2)}, ..., A^{(n-1)}$ be the submatrices generated by the Gaussian elimination process. Show, in the case that no pivoting is applied, that if $A^{(i)}$ is strictly diagonally dominant then also $A^{(i+1)}$ is. What does this mean for the stable solution of the associated system?

d. Describe the application of the Jacobi method to the above system.

e. Show that the Jacobi method converges for the above system and give an estimate for the convergence rate.

Problem 4

Assume A is an $n \times n$ full symmetric matrix. Describe the ingredients for an efficient computation of all eigenvalues of this matrix by the QR algorithm.